AD POLOSA — SAPIENZA UNIVERSITY OF ROME

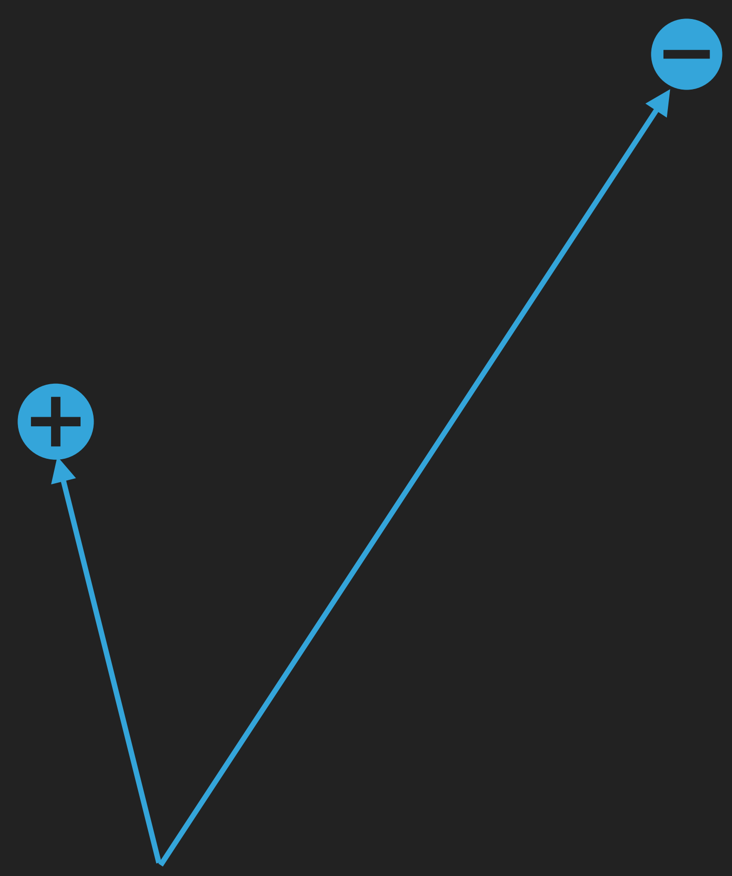
SYMMETRIES IN PHYSICS

A BRIEF ACCOUNT

"MATHEMATICS IS THAT PART OF PHYSICS IN WHICH EXPERIMENTS ARE LESS EXPENSIVE"

V.I. Arnold

Assumption: *the absolute position is non-observable*.



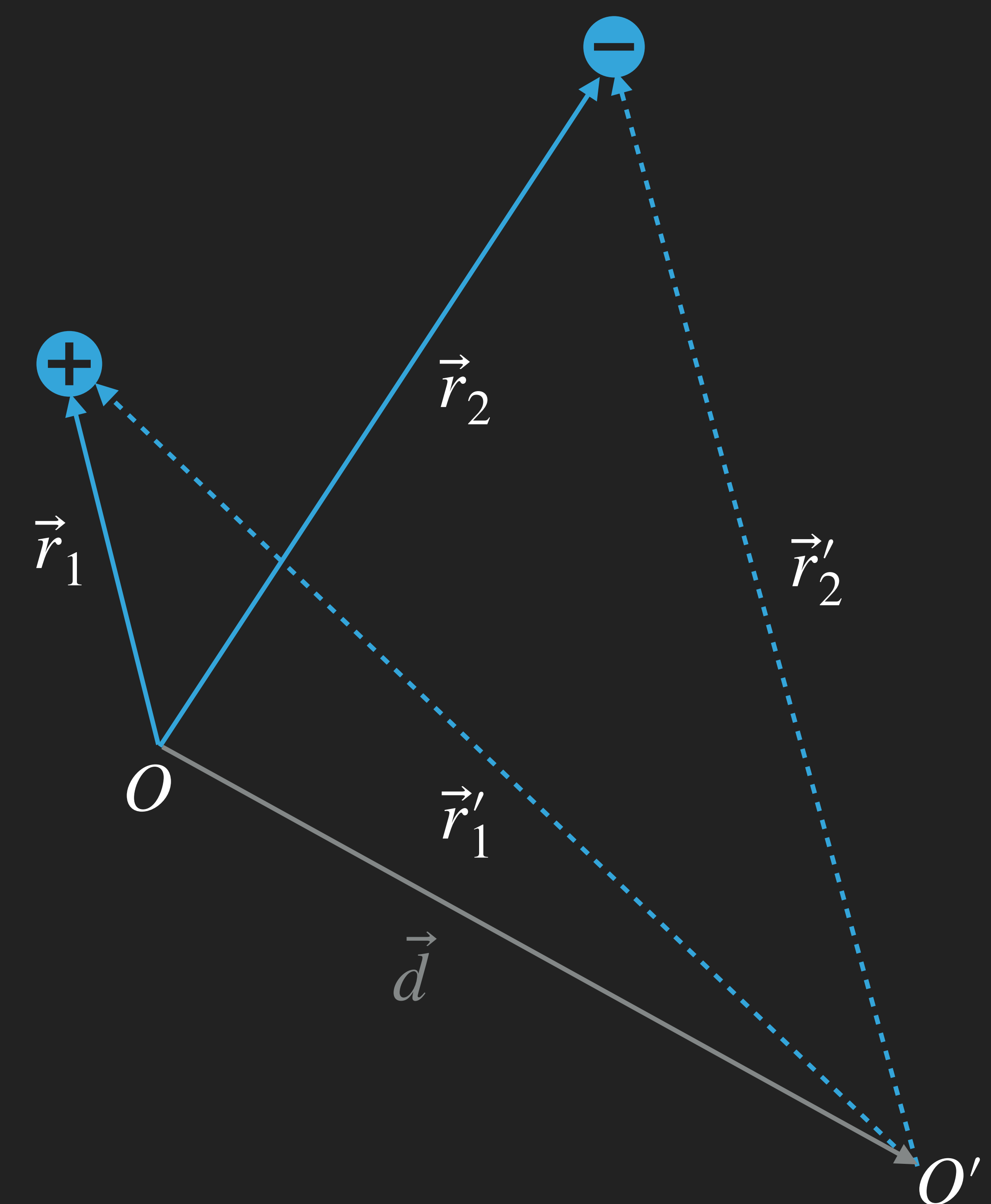
O is arbitrary

V = Interaction energy independent of O.

$\Rightarrow V$ invariant under space-translations

$\Rightarrow V = V(\vec{r}_1 - \vec{r}_2)$

$$V(\vec{r}_1 - \vec{r}_2) = V(\vec{d} + \vec{r}_1' - (\vec{d} + \vec{r}_2')) = V(\vec{r}_1' - \vec{r}_2')$$

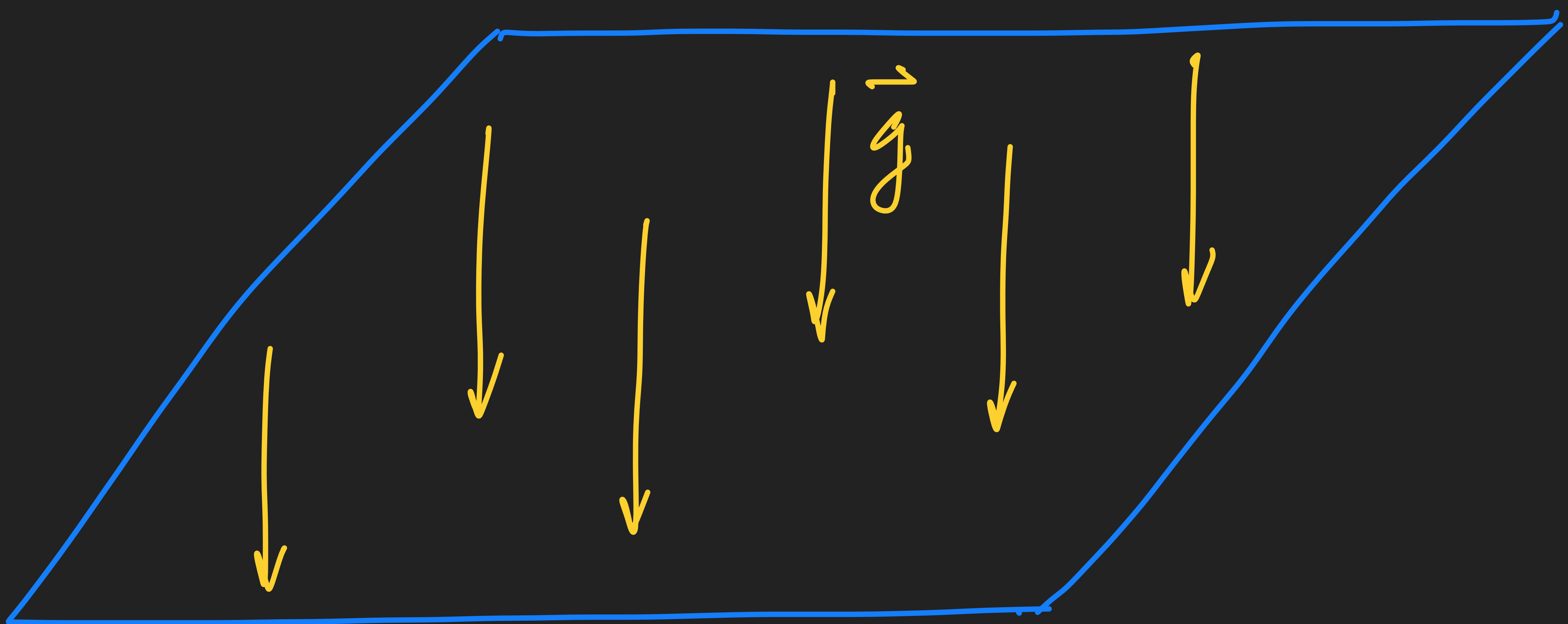


$$\frac{d\vec{P}}{dt} = -(\vec{\nabla}_1 + \vec{\nabla}_2)V = 0 \Rightarrow \vec{P} = \text{const.}$$

This means: invariance under space translations ⇒ conservation of total momentum.

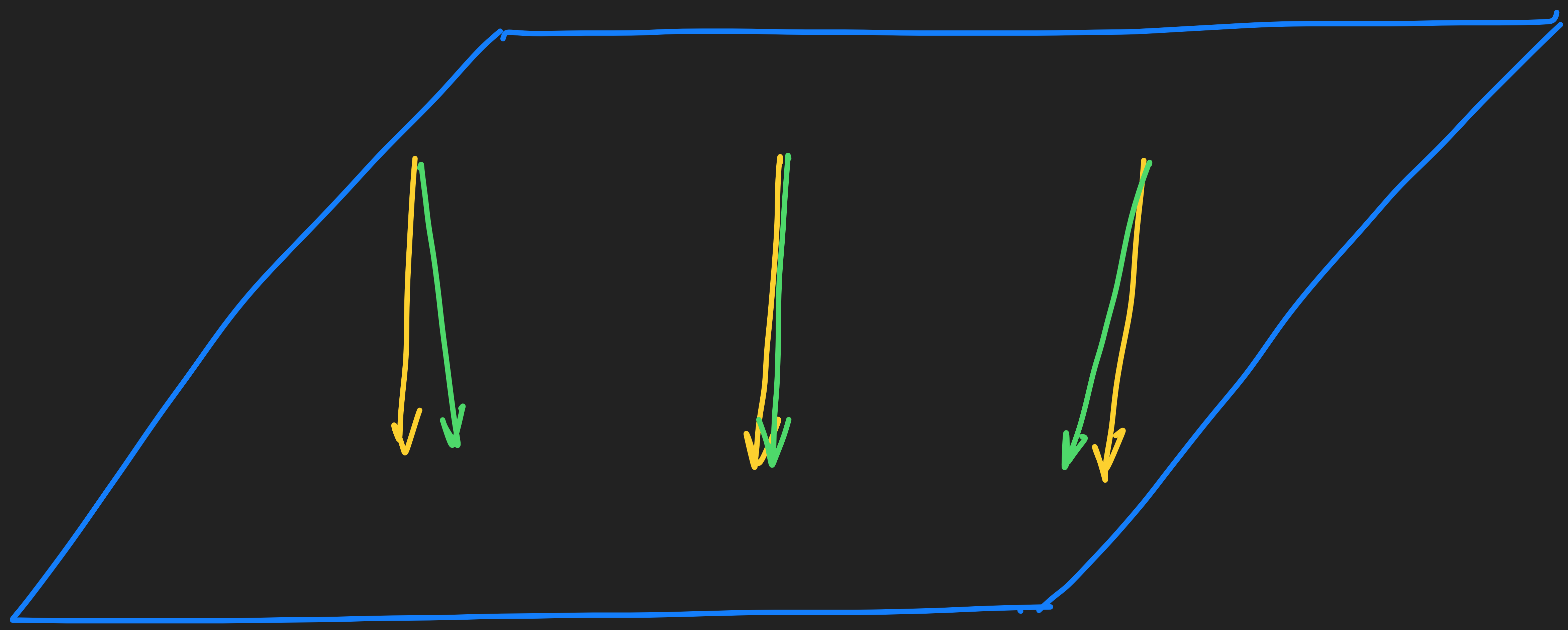
Non-Observables	Symmetry Transformation	Conservation Law
Absolute Spatial Position	$\vec{r} \rightarrow \vec{r} + \vec{d}$	Momentum
Absolute Spatial Direction	$r_i \rightarrow R_{ij} r_j$	Angular Momentum
Absolute Velocity	$x_\mu \rightarrow L_{\mu\nu} x_\nu$	Generators of Lorentz Group
Absolute Right/Left	$\vec{r} \rightarrow -\vec{r}$	Parity
Absolute Sign of Charge	$e \rightarrow -e$	Charge Conjugation
Absolute Time	$t \rightarrow t + \delta$	Energy

The gravity acceleration $\vec{g}$, pointing down, is an 'emergent' notion. This situation is not symmetric at all.

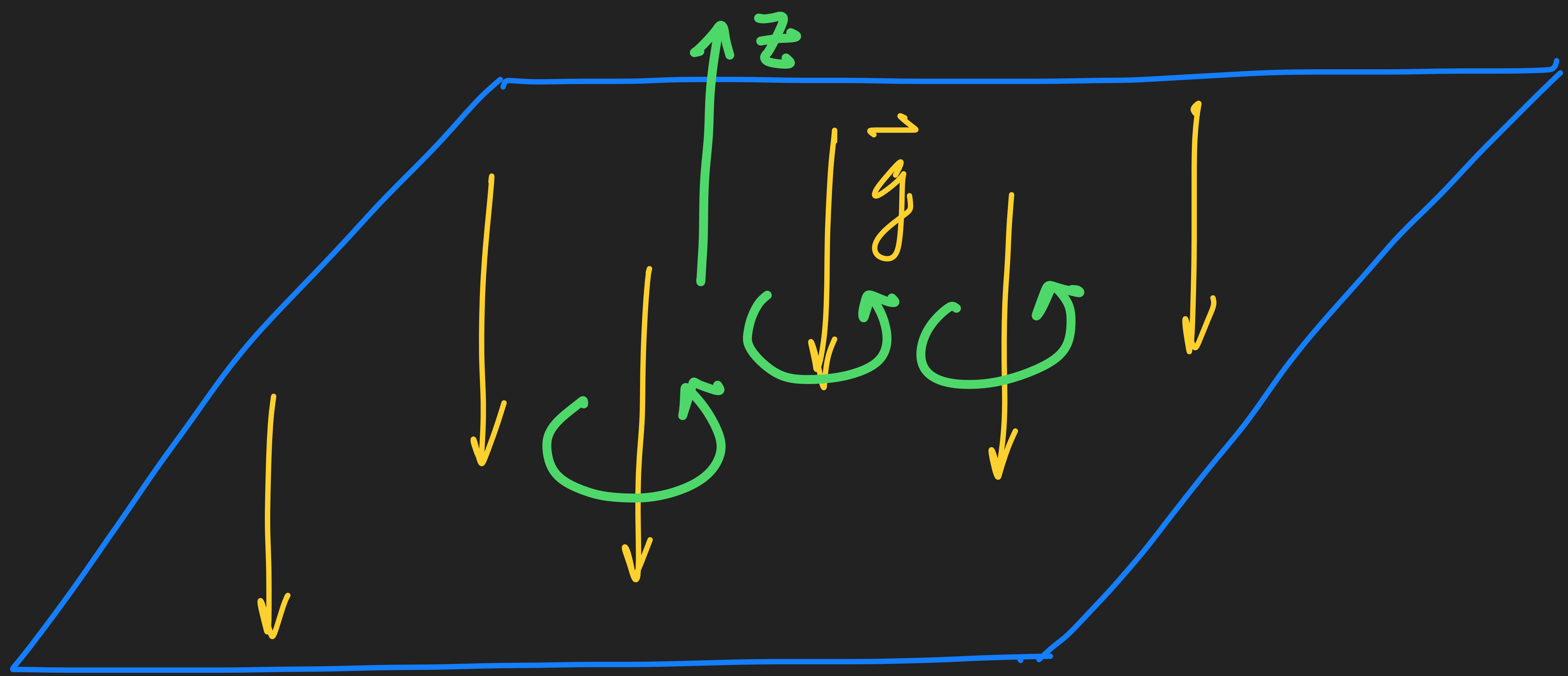


$|\vec{g}| = 9.8 \; m/sec^2$

Indeed gravity acceleration vectors are not parallel. They aim to the center of the earth.



Near the surface of earth we have $\overrightarrow{F} = m\vec{g}$, but $\vec{g}$ is fixed, and points down. Then $\overrightarrow{F}$ is *not invariant* under *all rotations*, but only under those rotations around the vertical axis.



However, Newton's law of gravitation is *rotation invariant*.

(here $m\vec{a}$ corresponds to a rotationally invariant term)

$$F = G\frac{m\,M}{r^2}$$

where r is the distance to the *center of the earth*.

This is a *symmetry* of a *fundamental law of physics*.

VECTORS AND ROTATIONS

A *vector* $\vec{F}$ is a set of numbers F_i which has given transformation properties under *rotations*

$$F_i \to F_i' = \sum_{j=1}^{3} R_{ij} F_j$$

$$R = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The equation $\vec{F} = m\vec{a}$ is a *relation between vectors*, both transforming with R. This equation is rotation invariant if both $\vec{a}, \vec{F}$ transform as a vectors (no $\vec{g}$ pointing up)

$$\vec{F}' = m\vec{a}'$$

RECOVERING ROTATION INVARIANCE

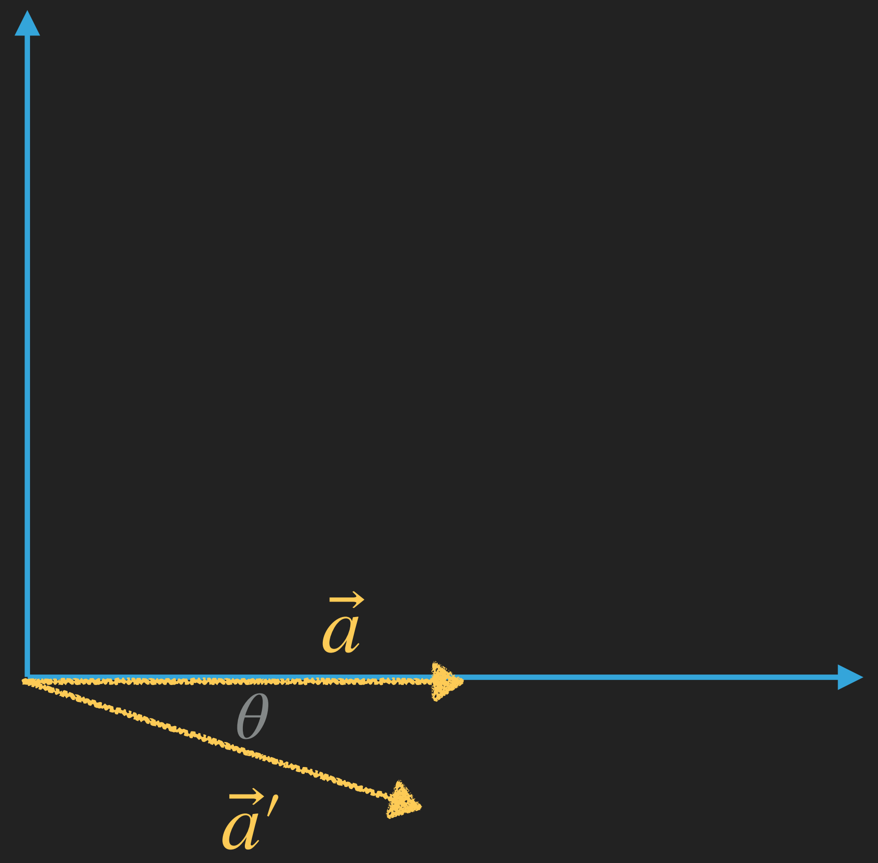
Invariance under rotations is *recovered* if in place of $\overrightarrow{F} = m\vec{g}$ we write $\overrightarrow{F}$ as the *gradient of the gravity potential*

$$m\vec{a} = -\overrightarrow{\nabla}\left(G\frac{mM}{r}\right)$$

where the gradient operator is

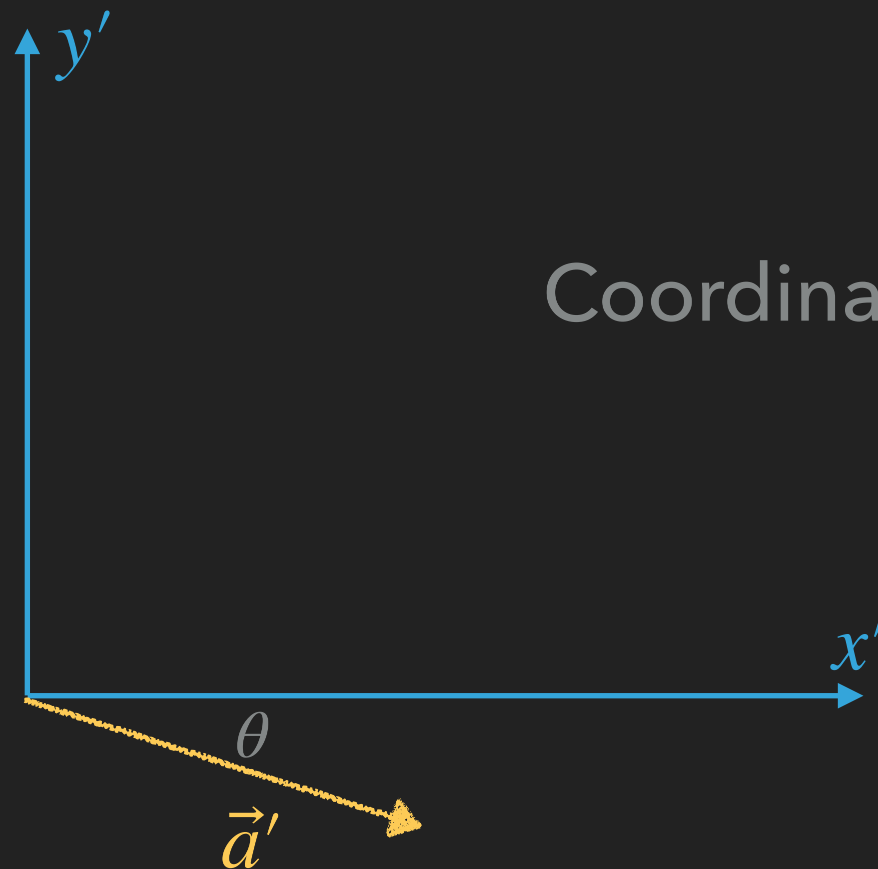
$$\overrightarrow{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

ROTATE THE VECTOR $\vec{a}$ INTO $\vec{a}'$



$$\underbrace{\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}}_{R(\theta)} \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{\vec{a}} = \underbrace{\begin{pmatrix} \cos\theta \\ -\sin\theta \end{pmatrix}}_{\vec{a}'}$$

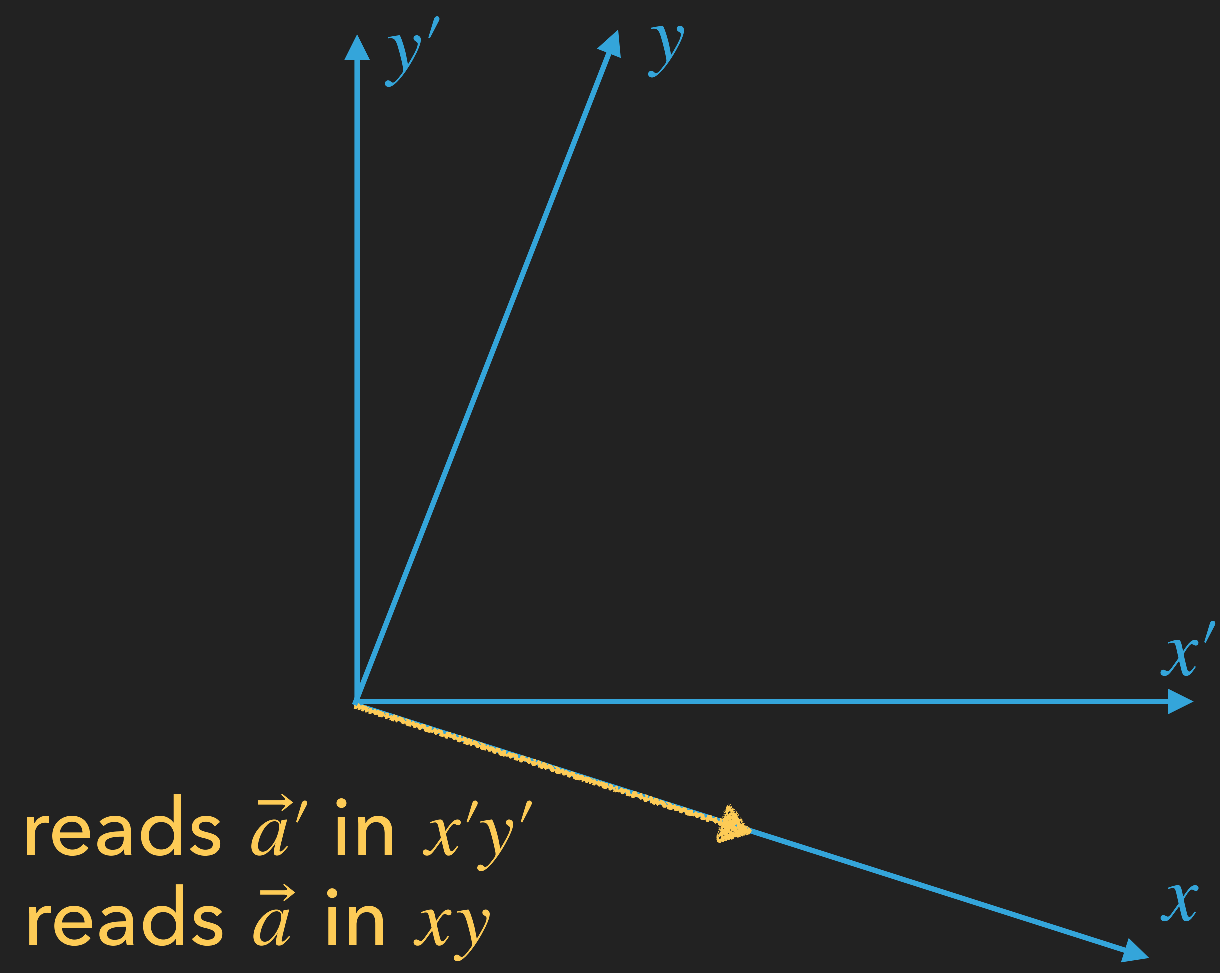
ROTATE THE VECTOR $\vec{a}$ INTO $\vec{a}'$



Coordinates of $\vec{a}'$ in $x'y'$

$$\vec{a}' = \begin{pmatrix} x' \\ y' \end{pmatrix} \equiv \begin{pmatrix} \cos\theta \\ -\sin\theta \end{pmatrix}$$

ROTATE THE FRAME xy IN THE FRAME $x'y'$



$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta \\ -\sin\theta \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$\tilde{R}(\theta)$ $\quad$ $\vec{a}'$ in $x'y'$ $\quad$ $\vec{a}$ in xy

The transpose works as the inverse transformation

$$\tilde{R}(\theta)$$

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$x = \cos\theta\, x' - \sin\theta\, y'$$
$$y = \sin\theta\, x' + \cos\theta\, y'$$

$$\frac{\partial}{\partial x'} = \frac{\partial x}{\partial x'}\frac{\partial}{\partial x} + \frac{\partial y}{\partial x'}\frac{\partial}{\partial y} = \cos\theta \frac{\partial}{\partial x} + \sin\theta \frac{\partial}{\partial y}$$
$$\frac{\partial}{\partial y'} = \frac{\partial x}{\partial y'}\frac{\partial}{\partial x} + \frac{\partial y}{\partial y'}\frac{\partial}{\partial y} = -\sin\theta \frac{\partial}{\partial x} + \cos\theta \frac{\partial}{\partial y}$$

Using the chain rule for derivatives we got

$$\underbrace{\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}}_{R} \begin{pmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial}{\partial x'} \\ \dfrac{\partial}{\partial y'} \end{pmatrix}$$

which is the *same* matrix R rotating $\vec{a}$ – the extension to three dimensions follows.

Apply the same R to both sides of the equation for the gravity acceleration

$$m\, R_{ij}\, a_j = -\, R_{ij}\, \nabla_j \left(G \frac{m\, M}{r} \right)$$

$$m\, \vec{a}' = -\, \overrightarrow{\nabla}' \left(G \frac{m\, M}{r} \right)$$

Notice that the *distance r is invariant under rotation* being

(scalar product) $r^2 = \vec{r} \cdot \vec{r} = \sum_i r_i r_i$

$$r'^2 = \sum_i \sum_j R_{ij}\, r_j \sum_k R_{ik}\, r_k = \sum_{i,j,k} r_j\, (\tilde{R})_{ji} R_{ik}\, r_k = r^2$$

The equation for the gravity acceleration reads the same in every rotated system.

$$m\,\vec{a}\,' = -\,\vec{\nabla}\,'\left(G\frac{m\,M}{r}\right)$$

ROTATION INVARIANCE OF MAXWELL EQUATIONS

THE 'VECTOR' PRODUCT OF TWO VECTORS

$$(\overrightarrow{A} \times \overrightarrow{B})_i \equiv \sum_{j,k=1}^{3} \epsilon_{ijk} A_j B_k$$

$$\epsilon_{123} = 1 \qquad \epsilon_{132} = -1$$

$$\epsilon_{231} = 1 \qquad \epsilon_{213} = -1$$

$$\epsilon_{312} = 1 \qquad \epsilon_{321} = -1$$

$$(\overrightarrow{A} \times \overrightarrow{B})_1 = (A_2 B_3 - A_3 B_2)$$

$$(\overrightarrow{A} \times \overrightarrow{B})_2 = (A_3 B_1 - A_1 B_3)$$

$$(\overrightarrow{A} \times \overrightarrow{B})_3 = (A_1 B_2 - A_2 B_1)$$

Maxwell's equations *are also invariant under rotations, e.g.*

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$R_{i'i} B_i = B'_{i'}$$

$$R_{i'i} (\epsilon_{ijk} \nabla_j E_k) = R_{i'i} \epsilon_{ijk} (\nabla'_{j'} R_{j'j}) (E'_{k'} R_{k'k}) = \vec{\nabla}' \times \vec{E}'$$

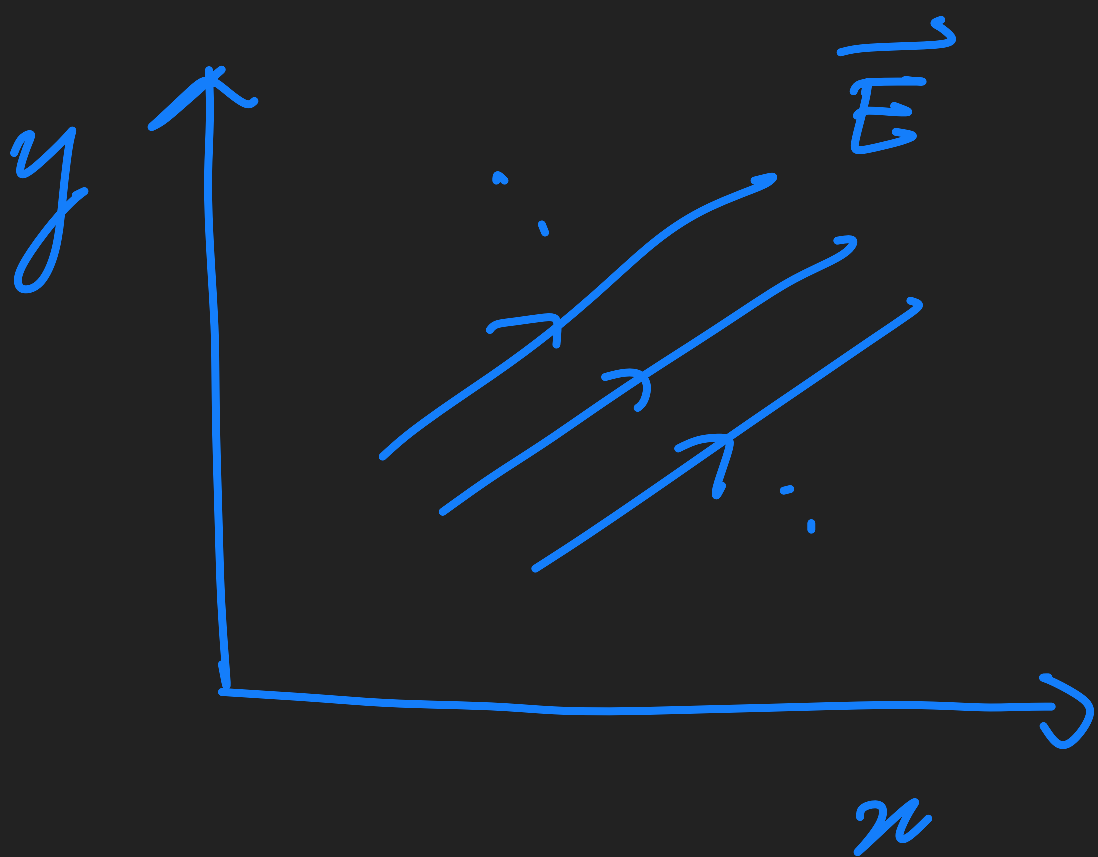
since $$\epsilon_{ijk} R_{i'i} R_{j'j} R_{k'k} = \det(R)\, \epsilon_{i'j'k'} = \epsilon_{i'j'k'}$$

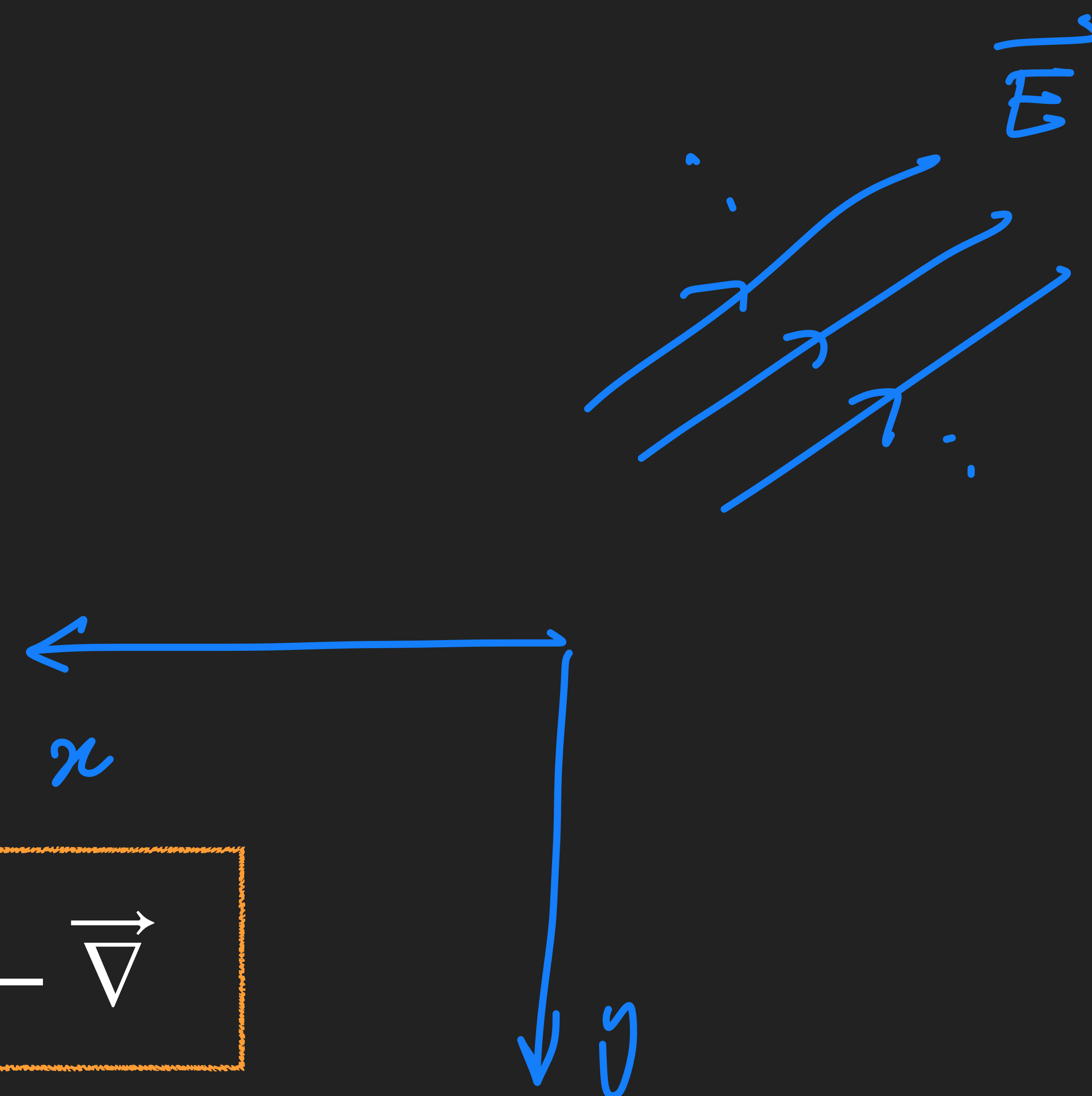
$$\vec{\nabla}' \times \vec{E}' = -\frac{\partial \vec{B}'}{\partial t}$$

PARITY

We can define *parity* P as

$$P\overrightarrow{E}(\vec{x}, t) = -\overrightarrow{E}(-\vec{x}, t)$$





$$P\overrightarrow{\nabla} = -\overrightarrow{\nabla}$$

This means that $\overrightarrow{\nabla} \times \overrightarrow{E}$ is left invariant by parity. If the magnetic field were invariant under parity (an axial-vector, parity even) we would say that the Maxwell equation at hand is also *parity-invariant*. Indeed

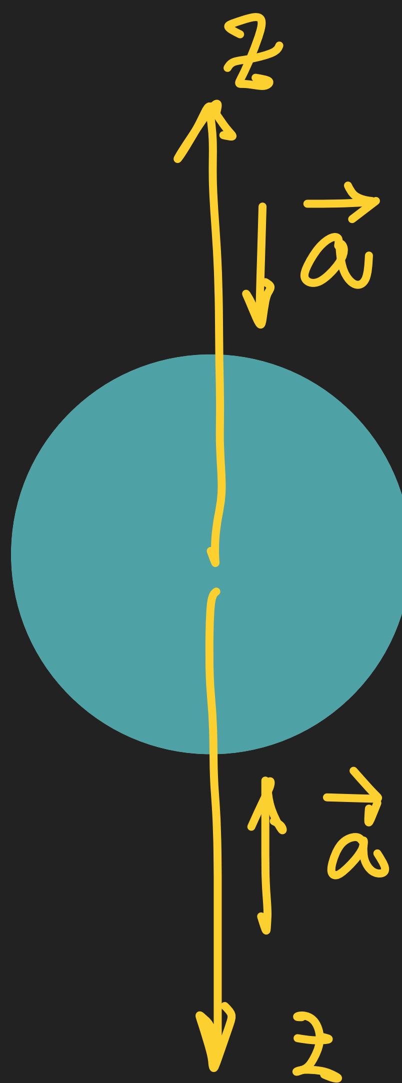
$$\overrightarrow{B} = \overrightarrow{\nabla} \times \overrightarrow{A}$$

is parity invariant, $\overrightarrow{A}$ being a vector – the 'vector potential'. Recall that $\overrightarrow{E} = -\partial \overrightarrow{A} / \partial t - \overrightarrow{\nabla} \phi$ and $\overrightarrow{E}$ is a vector (parity odd).

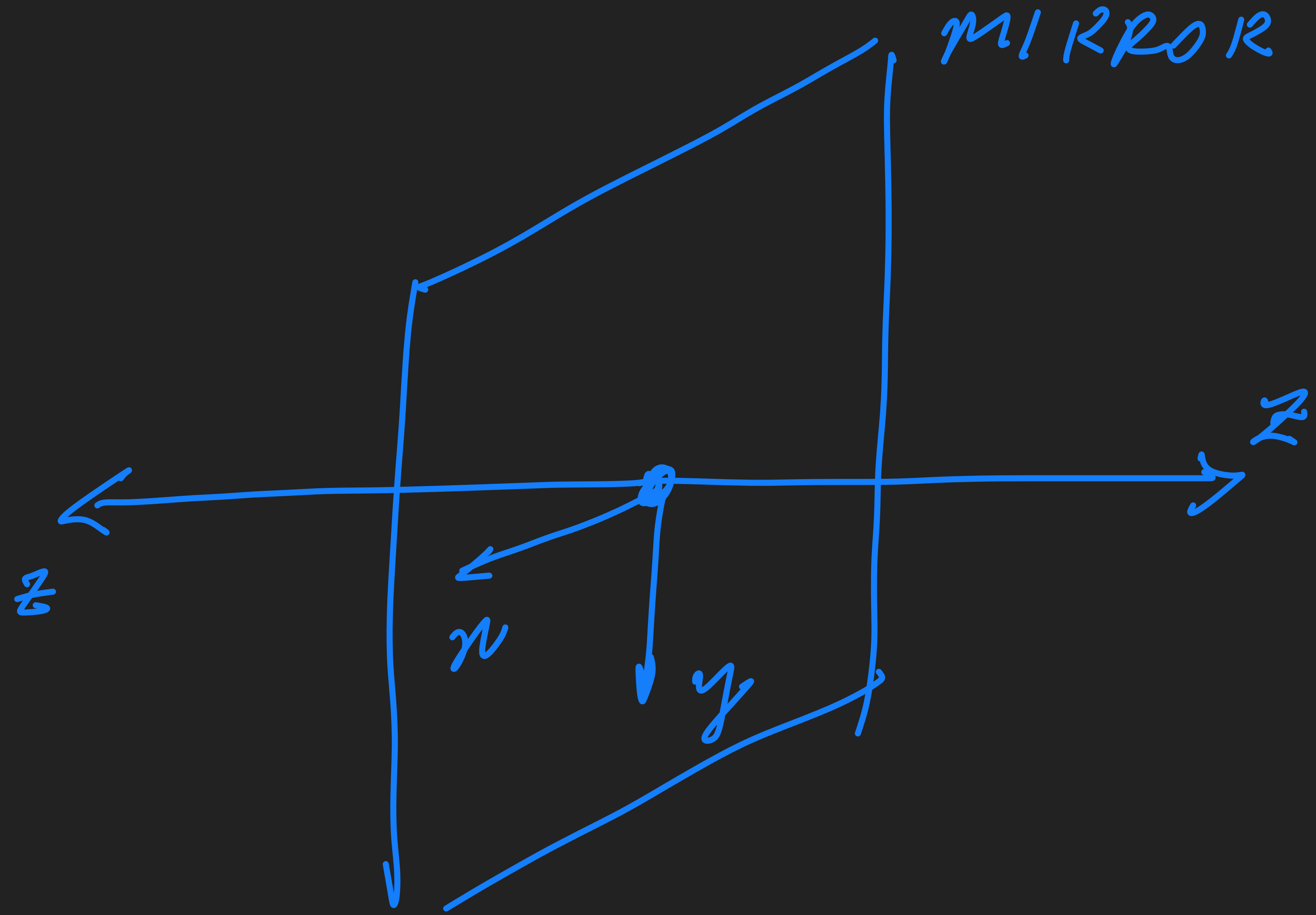
Also the other three M. equations are parity–invariant, as is the case for gravity law.

$$m\vec{a} = -\vec{\nabla}\left(G\frac{mM}{r}\right)$$

This equation is invariant under parity because $\vec{\nabla}$ flips under parity. N.B. $\vec{F} = m\vec{g}$ breaks parity (so as rotations).



It seems that the *fundamental laws* of physics *do not care about parity* (are parity-invariant), i.e. do not care about left and right, or up and down.



MIRROR: $z \longleftrightarrow -z$

$$P = \text{Mirror Symmetry} \times R_z(\pi)$$

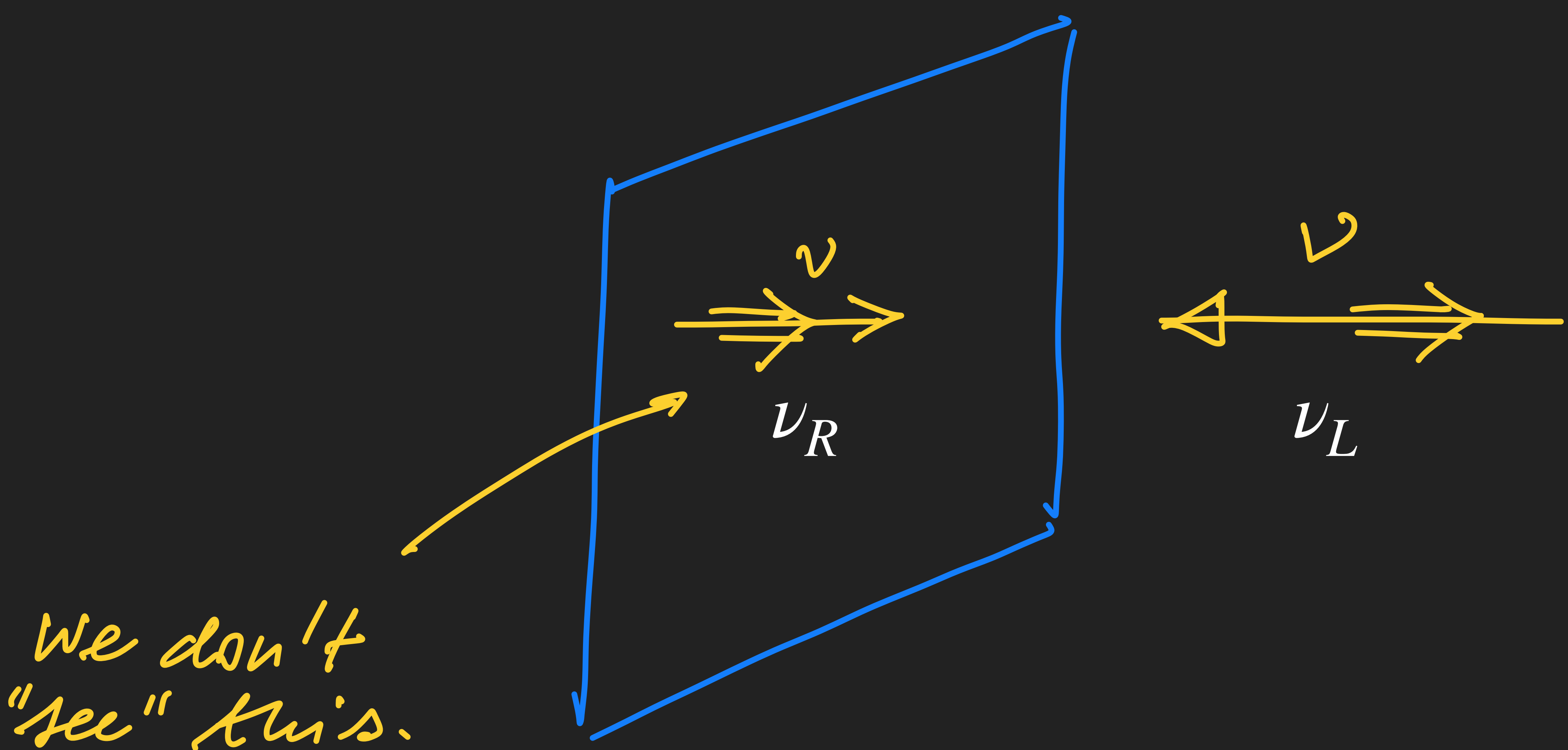
Since we have rotational invariance, the *mirror* is indeed performing a *parity operation*.

Is the world in the mirror allowed by physics?

Imagine a watch with the seconds hand only. Call it 'right' if the hand rotates clockwise (as it is usually the case!). Set the watch in front of a mirror and look in the mirror. You will see a 'left' watch.

PARITY VIOLATION

A *neutrino* with *left helicity* is sort of a left watch. Let it approach a mirror. The image in the mirror is a *right neutrino.* The neutrino is produced e.g. in nuclear β-decay. Are the laws governing neutrino production invariant under parity? **(CN Yang & TD Lee)**



WEAK INTERACTION CARES ABOUT LEFT & RIGHT!!

Neutrinos are neutral particles (almost massless). Despite its name, the `charge conjugation operation $C$`, makes changes on neutrinos: their handedness

$(C\nu_L)$ is the anti-$\nu = \bar{\nu}$

$(C\nu_L)$ is $R \Rightarrow \bar{\nu}_R$

However we still do not know if neutrinos are of the `Dirac-kind`, $\nu \neq \bar{\nu}$, or of the `Majorana-kind`, $\nu = \bar{\nu}$ (same particle in two handedness states). A non-zero mass Majorana neutrino can (rarely) flip its handedness.

Electrons etc., like neutrinos, have handedness too.

$$\pi^+ \to e^+(\nu_e)_L$$

almost forbidden because neutrino has to be L and anti-electron has to be R, e^+_R (electron, being almost massless wrt to pion, behaves like ν and should be L). On the other hand, conservation of angular momentum requires e^+_L and this *almost* kills the decay.

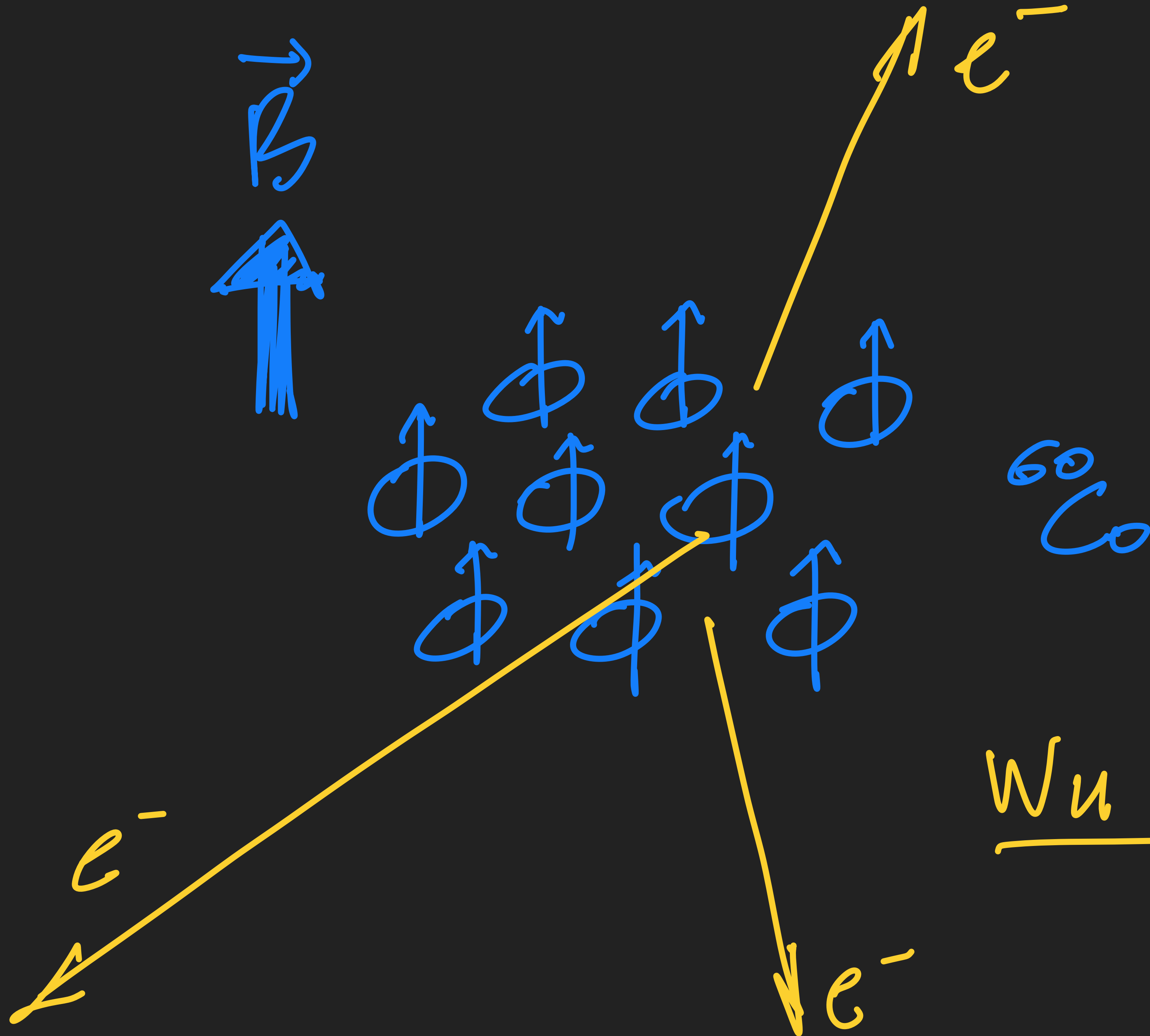
$$\pi^+ \to \mu^+(\nu_\mu)_L$$

this instead is allowed because μ is not much lighter than π

All this means we are giving an **absolute** value to the meaning of left/right.

“VIOLATION OF A SYMMETRY ARISES WHENEVER WHAT WAS THOUGHT TO BE NON-OBSERVABLE TURNS OUT TO BE OBSERVABLE!”

T.D. Lee



WU EXPERIMENT

UP-DOWN ASYMMETRY

It looks like the neutrino never had left/right symmetry in the first place. Maybe in the 'final' theory, handedness will be spontaneously broken as opposed to the contrived situation in which symmetry is broken explicitly.

food, ~ uniformly distributed along the circle – break the symmetry or starve…

CHARGE CONJUGATION

The rate counting below can differentiate e^- from e^+. Thus there is an absolute difference between the opposite signs of charges even though we are led to think to the sign of electric charge as merely conventional: electron is negative because we happened to assign + to the proton.

$$\frac{\Gamma(K_L^0 \to e^+ + \pi^- + \nu)}{\Gamma(K_L^0 \to e^- + \pi^+ + \bar{\nu})} \approx 1.0065$$

This means we are giving an **absolute** value to the meaning of +/- electric charge.

TIME REVERSAL

Another (discrete) symmetry is *time-reversal*

$$\vec{x} \to \vec{x} \qquad t \to -t$$

$$\vec{v} = \frac{d\vec{x}}{dt} \to -\vec{v}$$

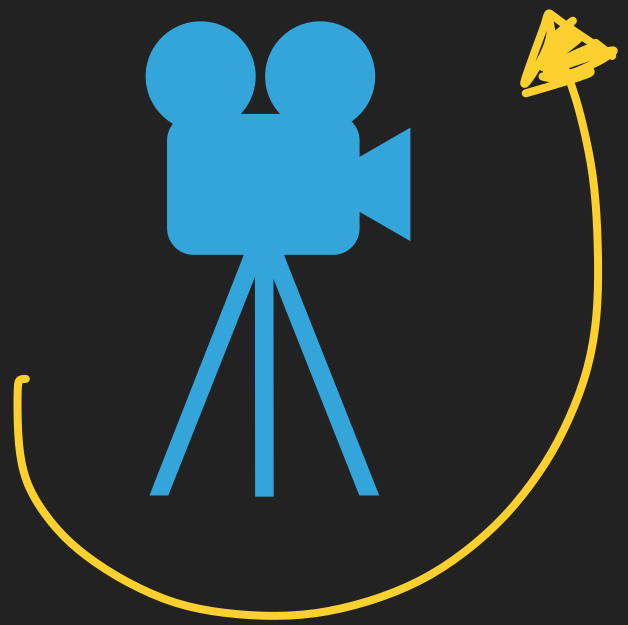
$$\vec{a} = \frac{d\vec{v}}{dt} \to \vec{a}$$

Consider Newton law $\overrightarrow{F} = m\vec{a}$ with $\overrightarrow{F}$ derivable from a potential $V(\vec{x})$ as $\overrightarrow{F} = -\overrightarrow{\nabla} V(\vec{x})$.

$$\boxed{m\vec{a} = -\overrightarrow{\nabla} V(\vec{x})}$$

Is *invariant* under *time-reversal (not in presence of friction).*

There are no mirrors for *time-reversal:* look at the movie backwards



If the fundamental laws of physics are invariant under time-reversal, the motion in the reversed movie is a *possible motion.*

Under time-reversal the charge density ρ does not change sign. Since $\vec{\nabla} \cdot \vec{E} = \rho$ then $\vec{E}(\vec{x}, t) \to \vec{E}(\vec{x}, -t)$.

On the other hand

$$\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}$$

Thus $\vec{A}(\vec{x}, t) \to -\vec{A}(\vec{x}, -t)$ and therefore $\vec{B}(\vec{x}, t) \to -\vec{B}(\vec{x}, -t)$ (since one can show that $\Box\phi = -\rho$ in the 'Lorenz gauge').

Thus

$$\boxed{\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}}$$

is *invariant* under *time-reversal!* (so are the other ones)

It seems that the *fundamental laws* of physics *do not care about time-reversal* (are time-reversal-invariant), i.e. the backwards movie is always possible.

However the physics of fundamental interactions is surprising again: there are $T-$violations, or $CP-$violations (CPT is conserved) in weak interactions!

These facts are the backbone of the *Standard Model* of particle physics.

In the symmetric phase, electrons, quarks and mediators in the Standard Model are considered massless. Upon spontaneous breaking of the electroweak symmetry they can acquire mass.

Massless quarks & leptons

$$\begin{pmatrix} u \\ d \end{pmatrix}_L \quad \begin{pmatrix} c \\ s \end{pmatrix}_L \quad \begin{pmatrix} t \\ b \end{pmatrix}_L$$

$$\begin{pmatrix} \nu_e \\ e \end{pmatrix}_L \quad \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}_L \quad \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix}_L$$

Massive & massless gauge vectors

$$W^\pm, Z^0 \qquad \gamma, g_1, \ldots, g_8$$

Higgs particle

$$H^0$$